Exam Sets & Numbers

(PART 2: INTEGERS AND MODULAR ARITHMETIC)
November 6th, 2024, 8:30am–10:30am,
Exam Hall 4 S1 - Y22.

Write your name on every sheet of paper that you intend to hand in.
Please provide **complete** *arguments for each of your answers.*
You may use a simple (not programmable) calculator during the exam.
This exam consists of 3 questions. You can score up to 6 points for each question, and you obtain 2 points for free.
In this way you will score in total between 2 and 20 points.

(1) In this exercise we let a, b be integers satisfying the property $5 \in \mathbb{Z} \cdot a + \mathbb{Z} \cdot b$.

(a) [2 points]. Show that if also $42 \in \mathbb{Z}\cdot a+\mathbb{Z}\cdot b$, then $\gcd(a, b) = 1$.

(b) [2 points]. Give an example of such a, b with $42 \notin \mathbb{Z} \cdot a + \mathbb{Z} \cdot b$ (prove that indeed your example has the desired properties!)

(c) [2 points]. Show that ($a = 1105$, $b = 2024$) provides an example of a pair as discussed here; so $5 \in \mathbb{Z} \cdot 1105 + \mathbb{Z} \cdot 2024$.

(2) For $k \in \mathbb{Z}_{\geq 1}$ consider the numbers $m_k := -1 + 2^k$. Three weeks ago (October 12th) it was discovered that $m_{136279841}$ is a prime number (the decimal expansion of it has 41024320 digits).

(a) [2 points]. Show that if $k|\ell$ for positive integers k, ℓ, then $m_k|m_\ell$.

(b) [2 points]. According to wikipedia, the last two decimal digits of $m_{136279841}$ are 51. Prove that this is indeed the case.

(c) [2 points]. Prove that if p is any prime number, then it follows that $p|(m_p - 1)$.

(3) This is an exercise about the integer $n = 42^{42}$.

(a) [2 points]. Explain why $n + 1$ is not a prime number.

(b) [2 points]. Show that $11|(n - 4)$.

(c) [2 points]. Suppose that $\bar{x} \in \mathbb{Z}/n\mathbb{Z}$ satisfies $\bar{5} \cdot \bar{x} = \bar{2}$. Prove that $\bar{x} \notin (\mathbb{Z}/n\mathbb{Z})^\times$.